Working Toward a Color Space Built on DE2000

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QuadTech
Problem to be Solved

To make a ramp that looks linear
Why is this a problem?

Is one of these half way between white and black?
Why is this a problem?

Isn’t 50% reflectance half way?
Why is this a problem?

- Human eye is nonlinear
- Very sensitive at low reflectance
- Not very sensitive at high reflectance
- Much effort has been put into “the formula”
History Lesson – Munsell Color Space

Albert Munsell
1858 - 1918
History Lesson – Munsell Color Space
History Lesson – Munsell Color Space

Munsell mixed paints to achieve perceptual linearity
The “1931 Standard Observer”
Tristimulus Values: X, Y, and Z

7.1 Calculation of tristimulus values

The CIE Standard (CIE, 1986a) on standard tristimulus values of a colour stimulus be value of the colour stimulus function \( \phi_\lambda(\lambda) \), functions and integrating each set of products the entire visible spectrum, 360 nm to 8
numerical summation at wavelength interval

\[
X = k \sum_\lambda \phi_\lambda(\lambda) \bar{x}(\lambda) \Delta \lambda \\
Y = k \sum_\lambda \phi_\lambda(\lambda) \bar{y}(\lambda) \Delta \lambda \\
Z = k \sum_\lambda \phi_\lambda(\lambda) \bar{z}(\lambda) \Delta \lambda
\]
Chromaticity Diagram

MacAdam Ellipses
Chromaticity Diagram

Chromaticity space is NOT perceptually linear!
TABLE 8.1
FAMILY TREE OF COLOR SCALES

- **ALL STIMULI**
  - CIE Y, x, y
    - 1931
  - Judd Maxwell Triangle 1935
  - MacAdam u, v
    - Diagram 1937
  - Breckenridge and Schaub
    - RUQS 1939

- **ALL OBJECTS**
  - Dominant Wavelength and Purity 1931
  - MacAdam Ellipses 1942
  - Friele 1961
  - FMC 1967
  - Hunter a, b
    - 1942

- **OPAQUE DIFFUSE OBJECTS**
  - Munsell 1929
  - Munsell Renotation 1943
  - Adams Chromatic Value 1942
  - Adams–Nickerson 1944, 1950
  - Glasser Cube Root 1958
  - Hunter R*, a, b
    - 1946
  - Hunter L*, a, b
    - 1958
  - CIE L*"a"*b*" 1964
  - CIE L*"u"*v* 1976

*The Measurement of Appearance, Richard Hunter, 1987*
Three-dimensional, approximately uniform, colour space produced by plotting in rectangular coordinates, $L^*$, $a^*$, $b^*$, quantities defined by the equations:

\[
L^* = 116 f(Y/Y_n)-16 \\
a^* = 500[f(X/X_n) - f(Y/Y_n)] \\
b^* = 200[f(Y/Y_n) - f(Z/Z_n)]
\]

where:

\[
f(X/X_n) = (X/X_n)^{1/3} \quad \text{if} \quad (X/X_n) > (24/116)^3 \\
f(X/X_n) = (841/108)(X/X_n)+16/116 \quad \text{if} \quad (X/X_n) \leq (24/116)^3
\]

and:

\[
f(Y/Y_n) = (Y/Y_n)^{1/3} \quad \text{if} \quad (Y/Y_n) > (24/116)^3 \\
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and:

\[
f(Z/Z_n) = (Z/Z_n)^{1/3} \quad \text{if} \quad (Z/Z_n) > (24/116)^3 \\
f(Z/Z_n) = (841/108)(Z/Z_n)+16/116 \quad \text{if} \quad (Z/Z_n) \leq (24/116)^3
\]

where $X, Y, Z$ are the tristimulus values of the test object colour stimulus considered and $X_n, Y_n, Z_n$ are the tristimulus values of a specified white object colour stimulus. In most cases, the specified white object colour stimulus should be light reflected from a perfect reflecting diffuser illuminated by the same light source as the test object. In this case, $X_n, Y_n, Z_n$ are the tristimulus values of the light source with $Y_n$ equal to 100.

CIE Colorimetry Technical Report 15:2004
People Involved in Standard

Members of the Technical Committee during the preparation of this report were:

<table>
<thead>
<tr>
<th>Name</th>
<th>Country</th>
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<tbody>
<tr>
<td>P.J. Alessi</td>
<td>USA</td>
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<tr>
<td>E.C. Carter</td>
<td>USA</td>
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Albuquerque, NM 2015
Color Difference, $\Delta E$

$\Delta E = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$

Albuquerque, NM 2015
Color Difference, $\Delta E$

$$\Delta E = \sqrt{(L_1 - L_2)^2 + (a_1 - a_2)^2 + (b_1 - b_2)^2}$$
Color Difference, $\Delta E$

Colour materials industries (textiles, paint, plastics, etc.) tended to adopt CIELAB and found significant evidence of non-uniformity of this colour space. McLaren introduced and McDonald further developed the idea of empirically correcting an approximate uniform colour space to improve the association between visual and numerical colour difference (McLaren, 1972; McDonald 1980a, 1980b, 1980c). The corrections generally are related to the location of a colour-difference pair in the colour space and the direction of difference between the colour-difference pair. This has led to the development of many CIELAB-based colour-difference models with a variety of empirical corrections fitted to specific sets of visual colour-difference data. Among these the CMC model has been widely adopted by the textile industry (Clarke, 1984).
\[ \Delta E_{\text{CMC}}(l : c) = cf \cdot \sqrt{\left( \frac{\Delta L^*}{l \cdot S_L} \right)^2 + \left( \frac{\Delta C^*}{c \cdot S_c} \right)^2 + \left( \frac{\Delta H^*}{S_H} \right)^2} \]
\[ \Delta E_{CMC} \]

\[ \Delta E_{CMC}(l : c) = cf \cdot \sqrt{\left( \frac{\Delta L^*}{l \cdot S_L} \right)^2 + \left( \frac{\Delta C^*}{c \cdot S_c} \right)^2 + \left( \frac{\Delta H^*}{S_H} \right)^2} \]

Albuquerque, NM 2015
\[ \Delta E_{CMC} (l : c) = cf \cdot \sqrt{\left( \frac{\Delta L^*}{l \cdot S_L} \right)^2 + \left( \frac{\Delta C^*}{c \cdot S_c} \right)^2 + \left( \frac{\Delta H^*}{S_H} \right)^2} \]

\[ S_L = \frac{0.040975 \cdot L^*}{(1 + 0.01765 \cdot L^*)} \quad \text{for} \quad L^* \geq 16 \]

\[ S_L = 0.511, \quad \text{for} \quad L^* < 16 \]

\[ S_C = \frac{0.0638 \cdot C^*}{(1 + 0.0131 \cdot C^*)} + 0.638 \]

\[ S_H = S_C (T \cdot f + 1 - f) \]

\[ f = \left\{ \frac{(C^*)^4}{(C^*)^4 + 1900} \right\}^{1/2} \]

\[ T = 0.56 + 10.2 \cos (h + 168^\circ), \quad \text{if} \quad 164^\circ < h < 345^\circ \]

\[ T = 0.36 + 10.4 \cos (h + 35^\circ), \quad \text{else} \]

ASTM D2244-07
McLaren-Style Corrections

- $\Delta E_{\text{JCP79}}$ – McDonald (1979)
- $\Delta E_{\text{CMC}}$ – McDonald, Clark, and Rigg (1986)
- $\Delta E_{\text{BFD}}$ – Luo and Rigg (1987)
- $\Delta E_{\text{94}}$ – CIE (1994)
- $\Delta E_{\text{LCD}}$ – Kim (1997)
- $\Delta E_{\text{00}}$ – CIE (2000)
$\Delta E_{00}$ Formula

$$\Delta E_{00} = \left[ \left( \frac{\Delta L'}{k_L S_L} \right)^2 + \left( \frac{\Delta C'}{k_C S_C} \right)^2 + \left( \frac{\Delta H'}{k_H S_H} \right)^2 + R_T \left( \frac{\Delta C'}{k_C S_C} \right) \left( \frac{\Delta H'}{k_H S_H} \right) \right]^{0.5}$$
\[
\Delta E_{00} = \sqrt{\left(\frac{\Delta L'}{k_Ls_L}\right)^2 + \left(\frac{\Delta C'}{k_Cs_C}\right)^2 + \left(\frac{\Delta H'}{k_Hs_H}\right)^2 + R_T \left(\frac{\Delta C'}{k_Cs_C}\right) \left(\frac{\Delta H'}{k_Hs_H}\right)}
\]

(B.1)

where

\(\Delta L'\) is the transformed lightness difference between specimens 1 and 2, see Equation (B.2);

\(\Delta C'\) is the transformed chroma difference between specimens 1 and 2;

\(\Delta H'\) is the transformed hue difference between specimens 1 and 2;

\(R_T\) is the rotation function, see Equation (B.11);

\(k_L, k_C\) and \(k_H\) are the parametric factors for variation in the experimental conditions;

\(s_L, s_C\) and \(s_H\) are the weighting functions, see Equations (B.7) to (B.9).

First, a localized modification of the scaling along the \(a^*\) axis is made which is most significant for colours at low chroma.

\[L' = L^*\]  \hspace{1cm} (B.2)

\[a' = a^*(1 + G)\]  \hspace{1cm} (B.3)

\[b' = b^*\]  \hspace{1cm} (B.4)
with

\[ G = 0.5 \left\{ \sqrt{C'_{ab}^*} - \sqrt{\frac{C'_{ab}^*}{C'_{ab}^* + 25}} \right\} \] (B.5)

and

\[ C'_{ab} = 0.5(C'_{ab1}^* + C'_{ab2}^*) \] (B.6)

where

- \( L' \) is the transformed lightness;
- \( a' \) is the transformed \( a^* \) (red-green opponent) co-ordinate;
- \( b' \) is the transformed \( b^* \) (yellow-blue) co-ordinate;
- \( G \) is a quantity that depends on the mean chroma of specimens 1 and 2.
The transformed $L'$, $a'$, $b'$ values are then used to calculate hue angle, chroma and lightness. These new quantities are designated by a prime mark. With these results the weighting functions and the rotation function are determined using the following equations:

$$S_L = 1 + \frac{0.015}{\sqrt{20 + (\bar{L}' - 50)^2}}$$

(B.7)

$$S_C = 1 + 0.045 \bar{C}'$$

(B.8)

$$S_H = 1 + 0.015 \bar{H}'$$

(B.9)

with

$$\bar{I} = 1 + 0.17 \cos(\bar{h}' - 30°) + 0.24 \cos(2\bar{h}') - 0.32 \cos(3\bar{h}' + 6°) - 0.20 \cos(4\bar{h}' - 63°)$$

(B.10)

where $S_L$, $S_C$ and $S_H$ are the weighting functions;

$\bar{L}'$ is the mean of the transformed lightnesses for specimens 1 and 2;

$\bar{C}'$ is the mean of the transformed chromas for specimens 1 and 2;

$I$ is a quantity that depends on the mean of the transformed hue angles for specimens 1 and 2;

$\bar{H}'$ is the mean of the transformed hue angles for specimens 1 and 2.
Finally, the rotation function is calculated from the following equations:

\[ R_T = -R_C \sin(2\Delta\Theta) \quad \text{(B.11)} \]

with

\[ \Delta\Theta = 30^\circ \exp\left\{ -\frac{1}{275 + 25^7} \frac{1}{25^7} \right\} \quad \text{(B.12)} \]

and

\[ R_C = 2\sqrt{\frac{C'^7}{C'^7 + 25^7}} \quad \text{(B.13)} \]

where

- \( R_T \) is the rotation function;
- \( \Delta\Theta \) is the hue angle difference depending on the mean of the transformed hue angles for specimens 1 and 2;
- \( \overline{h} \) is the mean of the transformed hue angles for specimens 1 and 2;
- \( R_C \) is a quantity that depends on the mean of the transformed chromas for specimens 1 and 2;
- \( C' \) is the mean of the transformed chromas for specimens 1 and 2.
Parsimony

... even if you have 11,000 data points, you should be very careful about using eleven parameters in your regression.

http://johnthemathguy.blogspot.com/2012/07/when-regression-goes-bad.html
$2.0 \Delta E_{00}$ “Ellipses” Shown in L*a*b*
Uniform Color Space vs. Color Difference Formula

- The space itself is linear
- You can use Euclidean distance formula
- No corrections based on color
Uniform Color Spaces (Proposed)

- $\text{Lab}_{mg}$ (Colli, Gremmo, and Moniga, 1989)
- ATD (Guth, 1994)
- DCI-95 (Rohner and Rich, 1995)
- LLAB (Luo, Lo, and Kuo, 1995)
- ??? (Tremeau and Laget, 1995)
- CIECAM97 (CIE standard, 1997)
- RLAB (Fairchild, 1998)
- IPT (Ebner and Fairchild, 1998)
- L*’a*'b*’ (Thomsen, 1999)
- DIN99 (DIN standard 6176, 2000)
- CIECAM02 (CIE standard, 2002)
- QTD (Granger, 2008)
- $L^Ea^Eb^E$ (Berns, 2008)
- LAB2000 (Lissner and Urban, 2010)
Problem to be Solved

Right now, I’m just looking at $L^*$
Parsimony for \( L^* \)?

Three-dimensional, approximately uniform, colour space produced by plotting in rectangular coordinates, \( L^*, a^*, b^* \), quantities defined by the equations:

\[
L^* = 116 \left( \frac{Y}{Y_n} \right) - 16 \tag{8.3}
\]

\[
a^* = 500 \left[ f(X/X_n) - f(Y/Y_n) \right] \tag{8.4}
\]

\[
b^* = 200 \left[ f(Y/Y_n) - f(Z/Z_n) \right] \tag{8.5}
\]

where

\[
f(X/X_n) = \left( \frac{X}{X_n} \right)^{1/3} \quad \text{if} \quad (X/X_n) > \left( \frac{24}{116} \right)^3 \tag{8.6}
\]

\[
f(X/X_n) = \left( \frac{841}{108} \right) \left( \frac{X}{X_n} \right) + 16/116 \quad \text{if} \quad (X/X_n) \leq \left( \frac{24}{116} \right)^3 \tag{8.7}
\]

and

\[
f(Y/Y_n) = \left( \frac{Y}{Y_n} \right)^{1/3} \quad \text{if} \quad (Y/Y_n) > \left( \frac{24}{116} \right)^3 \tag{8.8}
\]

\[
f(Y/Y_n) = \left( \frac{841}{108} \right) \left( \frac{Y}{Y_n} \right) + 16/116 \quad \text{if} \quad (Y/Y_n) \leq \left( \frac{24}{116} \right)^3 \tag{8.9}
\]

and

\[
f(Z/Z_n) = \left( \frac{Z}{Z_n} \right)^{1/3} \quad \text{if} \quad (Z/Z_n) > \left( \frac{24}{116} \right)^3 \tag{8.10}
\]

\[
f(Z/Z_n) = \left( \frac{841}{108} \right) \left( \frac{Z}{Z_n} \right) + 16/116 \quad \text{if} \quad (Z/Z_n) \leq \left( \frac{24}{116} \right)^3 \tag{8.11}
\]

where \( X, Y, Z \) are the tristimulus values of the test object colour stimulus considered and \( X_n, Y_n, Z_n \) are the tristimulus values of a specified white object colour stimulus. In most cases, the specified white object colour stimulus should be light reflected from a perfect reflecting diffuser illuminated by the same light source as the test object. In this case, \( X_n, Y_n, Z_n \) are the tristimulus values of the light source with \( Y_n \) equal to 100. 

CIE Colorimetry Technical Report 15:2004
Parsimony for $L_{mg}$?

\[ L_{mg} = 34.7231 \times \frac{(0.01765L + \ln((0.040975L)))}{0.040975} \]

\[ L_{mg} = \frac{L^*}{0.511} \]

4

5

6

7

$L^* \geq 16$

$L^* < 16$
## Parsimony for $\Delta L$

<table>
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<tr>
<th>Color Difference Equation</th>
<th># of parameters for $\Delta L$</th>
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<tbody>
<tr>
<td>$\Delta E_{ab}$</td>
<td>3</td>
</tr>
<tr>
<td>$\Delta E_{CMC}$</td>
<td>6</td>
</tr>
<tr>
<td>$\Delta E_{94}$</td>
<td>3</td>
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<td>$\Delta E_{00}$</td>
<td>6</td>
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<tr>
<td><strong>Uniform Color Space</strong></td>
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<tr>
<td>$LAB_{mg}$</td>
<td>7</td>
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<tr>
<td>$L^{**}$ (Rohner and Rich)</td>
<td>5</td>
</tr>
<tr>
<td>$\Delta E_{99}$ (DIN 99)</td>
<td>5</td>
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</table>
Building \( L_{00} \)

\[
\begin{align*}
\{0, 0, 0\} & \rightarrow 1.0 \Delta E_{00} \\
\{1.734, 0, 0\} & \rightarrow 1.0 \Delta E_{00} \\
\{3.442, 0, 0\} & \rightarrow 1.0 \Delta E_{00} \\
\{5.124, 0, 0\} & \rightarrow 1.0 \Delta E_{00} \\
\ldots
\end{align*}
\]
Building $L_{00}$
Building $L_{00}$

The only thing worthwhile in this whole presentation

$$25 \log_e (20Y + 1)$$
Error in $L_{00}$
## Parsimony for $\Delta L$

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<td>5</td>
</tr>
<tr>
<td>$\Delta E_{99}$ (DIN 99)</td>
<td>5</td>
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<tr>
<td>$L_{00}$</td>
<td>2</td>
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This general formula is nothing new!

- Delboeuf Equation (1872)
  \[ D = 10 - 6.1723 \log_{10} (40.7h + 1) \]

- Richter’s equation (1953)
  \[ V = 6.1723 \log_{10} (40.7Y + 1) \]
Applying the Equation to $a_{00}b_{00}$
Progress so far

- \( L_{00} \) – Looks good
- \( a_{00}, b_{00} \) – same approach doesn’t help much
Stay tuned…

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